COMBINATORIAL MODEL CATEGORIES

Brad Drew

Fix a Grothendieck universe $\mathcal{U}$ such that $\mathcal{Z} \in \mathcal{U}$. Unless otherwise indicated, sets are presumed to be $\mathcal{U}$-small and categories to be locally $\mathcal{U}$-small.

1 Locally presentable categories

In this section, $\mathcal{C}$ denotes a fixed category, which is locally $\mathcal{U}$-small by convention. Our goal here is to define locally presentable categories, which are, very roughly, categories determined by colimits of nice diagrams in a prescribed $\mathcal{U}$-small full subcategory. Under this interpretation as being completely determined by a $\mathcal{U}$-small category, it is not surprising that the theory of localizations of such categories is more tractable from a set-theoretic perspective than the general case. We only present the definition here. The reader will find detailed exposition of the theory of locally presentable categories in [AR94], [Bar10], [Bek00], [Bor94] Chapter 5 and [Lur09] §5.

Definition 1.1. A cardinal $\kappa$ is regular if, for each set $A$ of cardinal $< \kappa$ and each family of sets $\{X_a\}_{a \in A}$ indexed by $A$ such that $\text{card}(X_a) < \kappa$ for each $a \in A$, the union $\bigcup_{a \in A} X_a$ is of cardinal $\alpha \in A$.

Example 1.2.
(i) The cardinal $\aleph_0$ is regular: finite unions of finite sets are finite sets.
(ii) The axiom of choice implies that any successor cardinal is regular.
(iii) The cardinal $\aleph_\omega := \bigcup_{n \in \mathbb{Z}_{\geq 0}} \aleph_n$ is evidently not regular, since $\text{card}(\mathbb{Z}_{\geq 0}) < \aleph_\omega$.

Definition 1.3. Let $\kappa$ be a regular cardinal.

(i) A $\kappa$-directed set is a partially ordered set $A$ such that every subset $I \subseteq A$ of cardinal $< \kappa$ admits an upper bound. For example, $(\mathbb{Z}_{\geq 0}, \geq)$ is an $\aleph_0$-filtered set. Traditionally, we speak simply of “directed sets” when $\kappa = \aleph_0$. A $\kappa$-directed colimit in $\mathcal{C}$ is the colimit of a diagram indexed by a $\kappa$-directed set.

(ii) Assume $\mathcal{C}$ admits $\kappa$-directed colimits. We say that $X \in \text{ob}(\mathcal{C})$ is $\kappa$-presentable if $\text{mor}_\mathcal{C}(X, -): \mathcal{C} \to \text{Set}$ preserves $\kappa$-directed $\mathcal{U}$-colimits.

(iii) We say $\mathcal{C}$ is $\kappa$-accessible if it admits $\kappa$-directed $\mathcal{U}$-colimits and a set $\mathcal{S}$ of $\kappa$-presentable objects such that each $X \in \text{ob}(\mathcal{C})$ is the colimit of a $\kappa$-small $\kappa$-directed diagram in the full subcategory spanned by $\mathcal{S}$. We say that $\mathcal{C}$ is accessible if it is $\kappa$-accessible for some regular cardinal $\kappa$. Essentially by definition, a category is $\kappa$-accessible if and only if it is of the form $\text{Ind}_\kappa(\mathcal{C}_0)$ for some $\kappa$-small category $\mathcal{C}_0$.

(iv) We say $\mathcal{C}$ is locally $\kappa$-presentable (resp. locally presentable) if it is $\kappa$-accessible (resp. accessible) and $\mathcal{U}$-cocomplete. It can be shown that any locally presentable category is also $\mathcal{U}$-complete. Grothendieck topoi and Grothendieck Abelian categories are examples of locally presentable categories, as are categories of presheaves on a $\mathcal{U}$-small category with values in any locally presentable category.

Remark 1.4. If $\mathcal{C}$ is $\kappa$-locally presentable for some $\kappa$, then each $X \in \text{ob}(\mathcal{C})$ is $\lambda$-presentable for some regular cardinal $\lambda$. Indeed, write $X = \text{colim}_{a \in A} G_a$ for some $\kappa$-directed diagram $(G_a)_{a \in A}$ of $\kappa$-presentable objects and let $\lambda$ be a regular cardinal $> \max(\kappa, \text{card}(A))$. Then, for any $\lambda$-directed diagram $(Y_\beta)_{\beta \in B}$ in $\mathcal{C}$, we have

$$\text{mor}_\mathcal{C}(X, \text{colim}_B Y_\beta) = \text{mor}_\mathcal{C}(\text{colim}_{a \in A} G_a, \text{colim}_B Y_\beta) \cong \text{lim}_{a \in A} \text{colim}_B \text{mor}_\mathcal{C}(G_a, Y_\beta) = \text{colim}_{\beta \in B} \text{colim}_{a \in A} \text{mor}_\mathcal{C}(G_a, Y_\beta) = \text{colim}_{\beta \in B} \text{mor}_\mathcal{C}(X, Y_\beta),$$

since $\lambda$-directed colimits in $\text{Set}$ preserve limits of size $< \lambda$.

2 Combinatorial model categories

In this section, $\mathcal{C}$ denotes a fixed locally presentable category.

Definition 2.1.
(i) A weak factorization system on $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of full subcategories of $\text{mor}(\mathcal{C})$ satisfying the following conditions:
(a) for each $f \in \text{mor}(\mathcal{C})$, there exist $l \in \mathcal{L}, r \in \mathcal{R}$ such that $f = rl$;
(b) $\mathcal{L}$ is the class of morphisms with the left lifting property with respect to $\mathcal{R}$;
(c) \( \mathcal{R} \) is the class of of morphisms with the right lifting property with respect to \( \mathcal{L} \).

(ii) A model structure on \( \mathcal{C} \) consists of the data of three classes \( (\text{cof}, \mathcal{W}, \text{fib}) \) of morphisms of \( \mathcal{C} \) such that:

(a) \( \mathcal{W} \) satisfies the two-out-of-three property, i.e. if \( f : X \to Y \) and \( g : Y \to Z \) are two morphisms in \( \mathcal{C} \) such that two elements of \( (f, g, gf) \) belong to \( \mathcal{W} \), then the third element of this set also belongs to \( \mathcal{W} \);

(b) the pairs \( (\text{cof} \cap \mathcal{W}, \text{fib}) \) and \( (\text{cof}, \mathcal{W} \cap \text{fib}) \) are both weak factorization systems on \( \mathcal{C} \). In this case, we call the elements of \( \text{cof} \) (resp. \( \mathcal{W}, \text{fib} \)) cofibrations (resp. weak equivalences, resp. fibrations) and the elements of \( \text{cof} \cap \mathcal{W} \) (resp. \( \mathcal{W} \cap \text{fib} \)) trivial cofibrations (resp. trivial fibrations).

Example 2.2. Model categories are useful because one can define a notion of homotopy between morphisms between bifibrant objects, i.e. objects \( X \) such that \( \mathcal{Q}_X \to X \) is a cofibration and \( X \to \ast \) a fibration, and the category whose objects are the bifibrant objects of \( \mathcal{C} \) and whose morphisms are the homotopy equivalence classes of morphisms in \( \mathcal{C} \) is equivalent to the category \( \mathcal{C}[\mathcal{W}^{-1}] \) obtained by formally inverting elements of \( \mathcal{W} \). In particular, \( \mathcal{C}[\mathcal{W}^{-1}] \) is locally \( \mathcal{U} \)-small—a property that fails miserably for arbitrary full subcategories \( \mathcal{W} \subseteq \mathcal{C}^{[1]} \).

If, as is often the case, we only use the model structure on \( \mathcal{C} \) to understand \( \mathcal{C}[\mathcal{W}^{-1}] \), then we can think of the model structure as playing the role of a basis for a vector space in linear algebra: there’s no canonical choice, so you are free to choose one that makes any particular calculation easier. If we try to pinpoint what an abstract vector space with no chosen coordinates is under this analogy, we are led to the notion of an \((\infty, 1)\)-category, as modeled, for example, by categories enriched in simplicial sets: see [3.1] for the tip of that iceberg.

Here are a couple garden-variety examples of model structures:

(i) Let \( \text{Set}_{\Delta} \) denote the category of simplicial sets. Let \( \mathcal{W} \) denote the class of weak homotopy equivalences, i.e. the class of morphisms \( f \) in \( \text{Set}_{\Delta} \) such that the geometric realization \( |f| \in \text{mor}(\text{Top}) \) induces an isomorphism on all homotopy objects. The class of cofibrations consists of the monomorphisms of \( \text{Set}_{\Delta} \) and the fibrations are those morphisms that satisfy the right lifting property with respect to the inclusions \( \Lambda^n_k \to \Delta^n \) for all \( n \in \mathbb{Z}_{\geq 0}, 0 \leq k \leq n \), where \( \Lambda^n_k \) denotes the \( k \)th horn obtained from \( \Delta^n \) by removing the interior and the \( k \)th face. These data form a model structure by [Qui67], Ch. 3, §3, Theorem 3). We could also deduce this from [2.5] below without much effort. The homotopy category of this model structure is the usual topological homotopy category.

(ii) Let \( A \) be a commutative ring. We then have an “A-linear” variant of the previous example. Specifically, let \( \text{Cplx}^{\leq 0}(A) \) denote the category of nonnegatively graded complexes of \( A \)-modules. Letting \( \mathcal{W} \) denote the class of quasi-isomorphisms and definition cofibrations and fibrations to be the classes of monomorphisms and morphisms with the right lifting property with respect to monomorphic quasi-isomorphisms, respectively, we also have a model structure whose homotopy category is the derived category \( \text{D}^{\leq 0}(A) \). The fibrant objects of this model structure are the degreewise injective complexes. Under the above analogy, this model structure is a basis adapted to calculations involving many right derived functors appearing in classical homological algebra. Dually, there is a model structure on \( \text{Cplx}^{\geq 0}(A) \) adapted to computing left derived functors.

Definition 2.3. Let \( I \subseteq \text{mor}(\mathcal{C}) \) be a set.

(i) We denote by \( \text{proj}(I) \) (resp. \( \text{inj}(I) \)) the class of all morphisms of \( \mathcal{C} \) with the left (resp. right) lifting property with respect to \( I \) and we set \( \text{cof}(I) := \text{proj}(\text{inj}(I)) \) and \( \text{fib}(I) := \text{inj}(\text{proj}(I)) \).

(ii) Let \( \text{pushout}(I) \) denote the class of morphisms \( f : C \to D \) in \( \mathcal{C} \) fitting into co-Cartesian squares of the form

\[
\begin{array}{ccc}
A & \xrightarrow{g} & C \\
\downarrow & & \downarrow f \\
B & \xrightarrow{f} & D
\end{array}
\]

such that \( g \in I \). A relative I-cell complex is a transfinite composition of morphisms in \text{pushout}(I). An I-cell complex is an object \( X \) of \( \mathcal{C} \) such that the canonical morphism \( \mathcal{Q}_X \to X \) is a relative I-cell complex. We denote by \( \text{cell}(I) \) the class of relative I-cell complexes in \( \mathcal{C} \).

Lemma 2.4 (Small object argument, [Rek00], 1.3, [Bar10], 1.25, [Lur09], A.1.2.5). Let \( \mathcal{C} \) be a locally presentable category and \( I := \{f_j : C_j \to D_j \mid j \in J\} \subseteq \text{mor}(\mathcal{C}) \) a set. Also, let \( [n] \) denote the totally ordered set \( \{0, 1, \ldots, n\} \) for each \( n \in \mathbb{Z}_{\geq 0} \) and \( \mathcal{C}^{[n]} \) the category of functors \( [n] \to \mathcal{C} \).

(i) There exists a functor \( T : \mathcal{C}^{[1]} \to \mathcal{C}^{[2]} \) sending \( f : X \to Z \) to a commutative triangle \( f = \pi_p \) with \( p \in \text{inj}(I) \) and \( i \in \text{cell}(I) \).

(ii) The class \( \text{cof}(I) \) consists of the all retracts of elements of \( \text{cell}(I) \), so, in particular, \( \text{cell}(I) \subseteq \text{cof}(I) \).

(iii) The pair \( \{\text{cof}(I), \text{inj}(I)\} \) is a weak factorization system on \( \mathcal{C} \).
Proof. In a locally presentable, each object is \( \kappa \)-presentable for some sufficiently large regular cardinal \( \kappa \). Let \( \kappa \) be a regular cardinal such that the domain and codomain of each element of \( I \) is \( \kappa \)-presentable and let \( f : X \to Z \) be a morphism of \( \mathcal{C} \). Define a functor \( \kappa \to \mathcal{C} \vert Z \) as follows: \( Y_0 := X \) and \( Y_\lambda := \operatorname{colim}_{\alpha < \kappa} Y_\alpha \) for each nonzero limit ordinal \( \lambda < \kappa \). If \( Y_\alpha \) has been defined for some \( \alpha < \kappa \), then we define \( Y_{\alpha+1} \) by the co-Cartesian square

\[
\begin{array}{ccc}
\bigsqcup_{j \in I, \eta \in F_j(Y_\alpha)} C_j & \to & Y_\alpha \\
\downarrow & & \downarrow \\
\bigsqcup_{j \in I, \eta \in F_j(Y_\alpha)} D_j & \to & Y_{\alpha+1}
\end{array}
\]

where \( F_j(Y_\alpha) := \operatorname{mor}_\mathcal{C}(D_j, Z) \times \operatorname{mor}_\mathcal{C}(C_j, Z) \operatorname{mor}_\mathcal{C}(C_j, Y_\alpha) \) for each \( j \in J \). Finally, define \( Y := \operatorname{colim}_{\alpha < \kappa} Y_\alpha \). If one stares at this construction long enough to see what is going on here, the lemma will prove itself. \( \square \)

**Theorem 2.5** (J. Smith, [Bar10 2.2], [Bek00 1.7]). Let \( \mathcal{C} \) be a locally presentable category, \( I, W \subseteq \mathcal{C}^{(1)} \) subcategories such that \( I \) is small and \( W \) is accessible and accessibly embedded, that is, there exists a regular cardinal \( \kappa \) such that \( W \) is accessible and accessibly embedded, \( W \subseteq \mathcal{C}^{(1)} \) are both \( \kappa \)-accessible and \( W \) is stable under \( \kappa \)-directed colimits in \( \mathcal{C}^{(1)} \). Assume further that:

(i) \( W \) satisfies the two-out-of-three property;
(ii) \( \operatorname{inj}(I) \subseteq W \); and
(iii) \( \operatorname{cof}(I) \cap W \) is stable under pushouts and transfinite composition.

Then there exists a small full subcategory \( J \subseteq \mathcal{C}^{(1)} \) such that \( \operatorname{cof}(J) = \operatorname{cof}(I) \cap W \) and \( \left( \operatorname{cof}(I), \operatorname{inj}(I) \right) \) and \( \left( \operatorname{cof}(J), \operatorname{inj}(\operatorname{cof}(J)) \right) \) are both weak factorization systems on \( \mathcal{C} \). In particular, these weak factorization systems make \( \mathcal{C} \) a model category.

**Proof.** By 2.4 it suffices to construct \( J \) such that \( \operatorname{cof}(J) = \operatorname{cof}(I) \cap W \). This is accomplished by the following lemmata. \( \square \)

**Lemma 2.6** ([Bar10 2.3], [Bek00 1.8]). With the notation and hypotheses of 2.5 assume \( J \subseteq \operatorname{cof}(I) \cap W \) satisfies the following condition: for any morphism \( \alpha : i \to w \) in \( \mathcal{C}^{(1)} \) with \( i \in I \) and \( w \in W \), there exists \( j \in J \) such that \( \alpha \) factors as \( i \to j \to w \). Then \( \operatorname{cof}(J) = \operatorname{cof}(I) \cap W \).

**Lemma 2.7** ([Bar10 2.4], [Bek00 1.9]). With the notation and hypotheses of 2.5 a set \( J \) satisfying the hypotheses of 2.6 exists.

**Definition 2.8.** A combinatorial model category is a locally presentable category \( \mathcal{C} \) equipped with a model structure arising from a set \( I \) and a class \( W \) as in 2.5.

### 3 Dwyer-Kan Simplicial Localization

Fix a model category \( \mathcal{C} \) with weak equivalences \( W \). Choose another Grothendieck universe \( \mathbb{U} \) such that \( \mathcal{C} \) is \( \mathbb{U} \)-small and let \( \textbf{Set} \) denote the category of \( \mathbb{U} \)-sets.

**Definition 3.1** ([DK80b], [DK80a]). Let \( X, Y \in \text{ob}(\mathcal{C}) \).

(i) A zigzag of length \( n \geq 0 \) from \( X \) to \( Y \) is a diagram

\[
\begin{array}{cccccc}
X & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & \cdots & \longrightarrow & C_{n-1} & \longrightarrow & Y
\end{array}
\]

in which the line segments denote arrows of \( \mathcal{C} \) either to the left or right, such that those pointing left belong to \( W \). A morphism of zigzags from \( X \) to \( Y \) is a commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & \cdots & \longrightarrow & C_{n-1} \\
C'_1 & \longrightarrow & C'_2 & \longrightarrow & C'_3 & \longrightarrow & \cdots & \longrightarrow & C'_{n-1}
\end{array}
\]

in which the vertical arrows belong to \( W \).
(ii) We say that two zigzags from X to Y are equivalent if they become equal after omitting identity morphisms and composing adjacent arrows facing the same direction. We let \( \text{zigzag}(X, Y) \) denote the category whose objects are equivalence classes of zigzags from X to Y and whose morphisms are equivalence classes of morphisms of zigzags.

(iii) The Dwyer-Kan simplicial localization \( \mathcal{L}^H \mathcal{L} \) of \( \mathcal{C} \) is the \( \text{Set}_\infty \)-category with the same objects as \( \mathcal{C} \) and with simplicial mapping spaces given by the nerve \( \mathbb{N}(\text{zigzag}(X, Y)) \). We let \( \text{map}_{\mathcal{E}, \mathcal{W}}(X, Y) \) denote the simplicial \( \mathcal{U} \)-set of morphisms from X to Y in \( \mathcal{L}^H \mathcal{L} \mathcal{C} \) and refer to it as the mapping space from X to Y in \( (\mathcal{C}, \mathcal{W}) \).

**Remark 3.2.** Essentially by definition, \( \tau_0 \text{map}_C(X, Y) \) is the \( \mathcal{U} \)-set of morphism \( \text{mor}_{\mathcal{C}[W^{-1}]}(X, Y) \), where \( \mathcal{C}[W^{-1}] \) is the locally \( \mathcal{U} \)-small category obtained by formally inverting elements of \( \mathcal{W} \). In particular, since \( \mathcal{C} \) is a model category, its homotopy category is locally \( \mathcal{U} \)-small, so, although the mapping spaces in \( \mathcal{L}^H \mathcal{L} \mathcal{C} \) are not \( \mathcal{U} \)-small, one can check that their homotopy objects are all \( \mathcal{U} \)-small. Also, \( \mathcal{C} \rightarrow \mathcal{L}^H \mathcal{L} \mathcal{C} \) is functorial in \( (\mathcal{C}, \mathcal{W}) \) and does not require \( \mathcal{C} \) to be a model category.

## 4 Left Bousfield Localization

Fix a model category \( \mathcal{C} \) and a set \( S \subseteq \text{mor}(\mathcal{C}) \).

**Definition 4.1.** A left Bousfield localization of \( \mathcal{C} \) with respect to \( S \) is an initial object in the category of left Quillen functors \( F : \mathcal{C} \rightarrow \mathcal{D} \) such that \( LF(s) \) is an isomorphism of \( \text{ho}(\mathcal{D}) \) for each \( s \in S \). In particular, as an initial object, it is unique up to isomorphism of model categories.

**Definition 4.2.** We say that \( X \in \text{ob}(\mathcal{C}) \) is \( S \)-local if, for each \( s \in S \), the induced morphism \( \text{map}_{\mathcal{C}, \mathcal{W}}(f, X) \) is a weak homotopy equivalence. We say that \( f \in \text{mor}(\mathcal{C}) \) is an \( S \)-local equivalence if, for each \( S \)-local object \( X \in \text{ob}(\mathcal{C}) \), the induced morphism \( \text{map}_{\mathcal{C}, \mathcal{W}}(f, X) \) is a weak homotopy equivalence.

**Lemma 4.3** ([Bar10, 4.6]). If \( \mathcal{C} \) is a combinatorial model category, then the class of \( S \)-local equivalences is an accessibly embedded accessible subcategory of \( \mathcal{C}^{[1]} \).

**Theorem 4.4** ([Bar10, 4.7]). If \( \mathcal{C} \) is a left proper\(^1\) combinatorial model category and \( S \subseteq \text{mor}(\mathcal{C}) \) a \( \mathcal{U} \)-set, then the left Bousfield localization \( \mathcal{L}_S \mathcal{C} \) exists. Also, the underlying category of \( \mathcal{L}_S \mathcal{C} \) is that of \( \mathcal{C} \); the two model structures have the same cofibrations; the fibrant objects of \( \mathcal{L}_S \mathcal{C} \) are the fibrant \( S \)-local objects of \( \mathcal{C} \); and the weak equivalences of \( \mathcal{L}_S \mathcal{C} \) are the \( S \)-local equivalences.

**Proof.** Let \( I \) be a set of generating cofibrations for \( \mathcal{C} \) and let \( \mathcal{W}_S \) denote the class of \( S \)-local equivalences, which, by 4.3, is an accessibly embedded accessible subcategory of \( \mathcal{C}^{[1]} \). Moreover, \( \text{cof}(I) \cap \mathcal{W}_S \) is stable under pushouts and transfinite compositions. This follows easily from the left properness of \( \mathcal{C} \) and the observation that \( \kappa \)-directed \( \mathcal{U} \)-colimits in the combinatorial model category \( \mathcal{C} \) are homotopy colimits for sufficiently large \( \kappa \) ([Bar10, 2.5]). We can therefore apply 2.5 to \( I \) and \( \mathcal{W}_S \).

**References**


---

\(^1\)A model category is left proper if weak equivalences in \( \mathcal{C} \) are stable under pushouts along cofibrations.